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On the Summation of Bessel Functions and Hankel Transforms

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1. INTRODUCTION

It is well known that the linear transformation formula for Jacobi's elliptic theta-function

$$\sum_{-\infty}^{+\infty} e^{n^2\pi i\tau + 2inz} = (-i\tau)^{-1/2} \sum_{-\infty}^{+\infty} e^{(n\pi - z)^2/\pi i\tau} \qquad (z \in \mathbb{C}, \operatorname{Im} \tau > 0) \qquad (1.1)$$

is equivalent to some important functional relations like, e.g., Poisson's summation formula or the functional equation for Riemann's zeta-function. Another remarkable example is the Doetsch–Kober summation formula for Bessel functions [2; 3, Part. III, pp. 238–41; 8], which has been generalized and transferred by A. Erdelyi [4] to the space of Hankel transforms

$$(H_\nu f)(s) = \int_I J_\nu(2(st)^{1/2}) f(t)\, dt \qquad (s \in \mathbb{R}^+), \qquad (1.2)$$

where $I \subseteq \mathbb{R}^+$ an interval, H_ν is a unitary self-adjoint operator on the Hilbert space of all functions $f \in L_2(I)$ and J_ν denotes the Bessel function of the first kind of order ν, Re $\nu > -\frac{1}{2}$. Now let $\Omega = (\mathbb{C}, +, \cdot)$ the vector space of all arithmetical functions $f\colon \mathbb{N} \to \mathbb{C}$ and define for g, $h \in \Omega$

$$s_q(n) = \sum_{d \mid g,\, c,\, d,\, (n,\, q)} g(d)\, h(q/d) \qquad (n, q \in \mathbb{N}). \qquad (1.3)$$

Then $s_q(n)$ is a periodic function of n with period q, has a finite Fourier expansion and represents a generalisation of Ramanujan's exponential sums $c_q(n)$ [1, pp. 160–4; 10]. In the present small note we combine arithmetical results on (1.3) with the Doetsch–Erdelyi integral transform

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methods [2, 4] in order to prove some new linear functional relations corresponding to (1.1) for series of Bessel–Schlömilch type (Section 2)

$$w(q, \nu, t) = \sum_{n \geq 1} s_q(n)\, n^{-\nu} J_\nu(2\pi n \sqrt{t}) \qquad (t \in \mathbb{R}^+), \tag{1.4}$$

and more generally of Hankel–Erdelyi type (Section 3)

$$w_f(q, \nu) = \sum_{n \geq 1} s_q(n)\, n^{-\nu} (H_\nu f)(\pi^2 n^2). \tag{1.5}$$

By special choice of the functions $g, h \in \Omega$ in (1.3) we obtain in Section 2 some important classical formulas for Schlömilch series, originally due to N. Nielsen [10, p. 336; 13, pp. 634–7], and reestablished by G. Doetsch [2], E. C. Titchmarsh [12], and recently by L. Lorch and P. Szego [9, p. 50]. An interesting application of special cases of (1.4) in mathematical physics; i.e., in the computation of lattice sums in ionic crystals is due to A. Hautot [6].

Finally by special choice of $H_\nu f$ in (1.5) and by use of some fractional integrals of Erdelyi–Kober type [5] we prove in Section 4 a general transformation formula which contains that for (1.4) as a limiting case.

Observe that series of type (1.5) with $H_\nu f$ replaced by inverse Mellin integral transforms play an important role in the Fourier analysis of arithmetical functions [7].

2. Hankel Kernels

The Dirichlet convolution of $g, h \in \Omega$ is defined by [1, p. 29]

$$D(n) = \sum_{d \mid n} g(d)\, h\left(\frac{n}{d}\right) \qquad (n \in \mathbb{N}). \tag{2.1}$$

Let

$$F(n) = \sum_{d \mid n} g(d)\, h\left(\frac{n}{d}\right) d^{-1} \qquad (n \in \mathbb{N}). \tag{2.2}$$

Since $s_q(n)$ is periodic mod q observe that the Dirichlet series

$$\sum_{n \geq 1} s_q(n)\, n^{-s} \tag{2.3}$$

converges absolutely for Re $s > 1$.

By application of the Laplace transform method of G. Doetsch [2] we then prove the following summation property for the Hankel kernel.

THEOREM A. *Let* $q\in\mathbb{N}$, $t\in\mathbb{R}^+$, $\operatorname{Re} \nu > -\frac{1}{2}$. *Then*

$$\frac{\pi^{\nu}t^{\nu/2}}{2\Gamma(\nu+1)}D(q)+\sum_{n\geqslant 1}s_q(n)\,n^{-\nu}J_\nu(2\pi n\sqrt{t})$$
$$=\frac{\pi^{\nu-1/2}t^{(1/2)(\nu-1)}}{2\Gamma(\nu+\frac{1}{2})}F(q)+\frac{\pi^{\nu-1/2}t^{-\nu/2}}{\Gamma(\nu+\frac{1}{2})}$$
$$\times\sum_{d\mid q}g(d)\,h\left(\frac{q}{d}\right)d^{-1}\sum_{\substack{1\leqslant m<d\sqrt{t}\\ m\in\mathbb{N}}}\left\{t-\left(\frac{m}{d}\right)^2\right\}^{\nu-1/2}. \tag{2.4}$$

Proof. For $\operatorname{Re} s\geqslant s_0$ let $\hat{f}(s)=L\{f(t)\}$ the one-sided Laplace transform for suitable restricted functions $f\colon\mathbb{R}^+\to\mathbb{C}$.

Using Hankel's generalisation of Weber's first exponential integral [13, p. 394]

$$\int_0^\infty J_\nu(at)\exp\{-p^2t^2\}\,t^{\nu+1}\,dt=\frac{a^\nu}{(2p^2)^{\nu+1}}\exp\left\{-\frac{a^2}{4p^2}\right\}$$

($\operatorname{Re}\nu>-1$, $|\arg p|<\pi/4$, $a\in\mathbb{C}$) we obtain the Laplace transform

$$L\{t^{\nu/2}J_\nu(n\sqrt{t})\}=\frac{1}{2^\nu s^{\nu+1}}n^\nu e^{-n^2/4s} \tag{2.5}$$

($t\in\mathbb{R}^+$, $s\in\mathbb{R}^+$, $\operatorname{Re}\nu>-\frac{1}{2}$). By (2.3) we have $\sum_{n\geqslant 1}|s_q(n)|\,e^{-n^2/4s}<\infty$. Thus term-wise application of (2.5) is allowed and we obtain

$$L\left\{2^\nu t^{\nu/2}\sum_{n\geqslant 1}s_q(n)\,n^{-\nu}J_\nu(n\sqrt{t})\right\}$$
$$=s^{-\nu-1}\sum_{d\mid q}g(d)\,h\left(\frac{q}{d}\right)\sum_{n\geqslant 1}\exp\{-n^2d^2/4s\} \tag{2.6}$$

($\operatorname{Re}\nu>-\frac{1}{2}$, $t\in\mathbb{R}^+$, $s\in\mathbb{R}^+$).

We now must consider the cases $\nu=0$ and $\operatorname{Re}\nu>0$ separately.

(i) $\nu=0$.

By (1.1) we obtain for $\operatorname{Re}\tau>0$ the linear transformation of Jacobi's theta function $\vartheta_3(0\mid\tau)$ in explicit form

$$1+2\sum_{n\geqslant 1}\exp\{-n^2\pi^2\tau\}=(\pi\tau)^{-1/2}\sum_{-\infty}^{+\infty}\exp\{-n^2\tau^{-1}\}. \tag{2.7}$$

By (2.1) and (2.2) we obtain from (2.6) and (2.7)

$$L\left\{\sum_{n\geqslant 1} s_q(n)\, J_0(n\sqrt{t})\right\}$$
$$=\sqrt{\frac{\pi}{s}}\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\left\{1+2\sum_{n\neq 1} e^{-n^2\pi^2 4sd^{-2}}\right\}-\frac{1}{2s}D(q). \quad (2.8)$$

Now since

$$\sqrt{\frac{\pi}{s}}\, e^{-\alpha s}=\int_\alpha^\infty (t-\alpha)^{-1/2}\, e^{-st}\, dt \qquad (\operatorname{Re} s>0,\ \alpha\geqslant 0)$$

we obtain with $L\{t^{-1/2}\}=\sqrt{\pi/s}$ for $m\in\mathbb{N}_0$

$$L\left\{\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\omega_m(t,d)\right\}=\sqrt{\frac{\pi}{s}}\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}e^{-4m^2\pi^2 sd^{-2}}, \quad (2.9)$$

where ω_m is defined by

$$\omega_m(t,d)=\begin{cases}0, & t\in[0,4\pi^2m^2d^{-2}]\\ (t-4m^2\pi^2d^{-2})^{-1/2}, & t>4m^2\pi^2d^{-2}.\end{cases} \quad (2.10)$$

Hence (2.8) becomes, observing $L\{1\}=s^{-1}$,

$$L\left\{\sum_{n\neq 1} s_q(n)\, J_0(n\sqrt{t})\right\}$$
$$=\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}L\left\{\omega_0(t,d)+2\sum_{m\geqslant 1}\omega_m(t,d)\right\}-\frac{1}{2}L\{1\}\, D(q). \quad (2.11)$$

The uniqueness theorem for Laplace transforms thus yields a.e., i.e., at all points of continuity

$$\sum_{n\geqslant 1} s_q(n)\, J_0(n\sqrt{t})$$
$$=\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\left\{t^{-1/2}+2\sum_{m\geqslant 1}\omega_m(t,d)\right\}-\frac{1}{2}D(q). \quad (2.12)$$

By (2.10) $\sum_{m\geqslant 1}\omega_m(t,d)$ reduces to a finite sum for fixed $t\in\mathbb{R}^+$. Thus (2.12) becomes for $t\in\mathbb{R}^+$

$$\sum_{n\geq 1} s_q(n)\, J_0(2\pi n\sqrt{t})$$

$$=\frac{t^{-1/2}}{2\pi} F(q)+\frac{1}{\pi}\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1} \sum_{\substack{1\leq m<d\sqrt{t}\\ m\in\mathbb{N}}} \left\{t-\left(\frac{m}{d}\right)\right\}^{-1/2}-\frac{1}{2}D(q).$$

This is Theorem A in the case $\nu=0$.

(ii) $\operatorname{Re}\nu>0$.

By case (i) and $L\{t^{\nu-1}/\Gamma(\nu)\}=s^{-\nu}$ ($\operatorname{Re}\nu>0$) the convolution theorem for Laplace transforms $L\{f*g\}=L\{f\}\cdot L\{g\}$ [3, Part II, p. 23] yields for (2.6)

$$2^{\nu}t^{\nu/2}\sum_{n\geq 1} s_q(n)\, n^{-\nu}J_0(n\sqrt{t})$$

$$=\frac{t^{\nu-1}}{\Gamma(\nu)} * \left[\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\left\{t^{-1/2}+2\sum_{m\geq 1}\omega_m(t,d)\right\}-\frac{1}{2}D(q)\right]. \tag{2.13}$$

Now $t^{\nu-1}*1=\nu^{-1}t^{\nu}$ and by (2.10) we obtain

$$t^{\nu-1} * \sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\omega_m(t,d)$$

$$=\begin{cases} 0, & t\in[0, 4m^2\pi^2d^{-2}] \\ \displaystyle\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\int_{4m^2\pi^2d^{-2}}^{t}\frac{(t-\tau)^{\nu-1}}{(\tau-4m^2\pi^2d^{-2})^{1/2}}\,d\tau, & t>4m^2\pi^2d^{-2}. \end{cases} \tag{2.14}$$

By means of Euler's beta-function $B(\alpha,\beta)=\int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1}\,d\tau$ ($\operatorname{Re}\alpha$, $\beta>-1$) (2.14) becomes for $t>4m^2\pi^2d^{-2}$

$$B\left(\nu,\frac{1}{2}\right)\sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1}\{t-4m^2\pi^2d^{-2}\}^{\nu-1/2}. \tag{2.15}$$

Inserting these results in (2.13) we obtain Theorem A in the case $\operatorname{Re}\nu>0$. The validity for $\operatorname{Re}\nu\geq 0$ follows by analytic continuation. But it is obvious that formula (2.4) also holds in the half-plane $\operatorname{Re}\nu>-\frac{1}{2}$.

We now discuss briefly an interesting special case of Theorem A: In (1.3) let $g(n)=n$ and $h(n)=\mu(n)$, where $\mu(\cdot)$ denotes the Möbius μ-function.

Then [1, Sect 8.3]

$$s_q(n)=c_q(n)=\sum_{d\mid(n,q)} d\mu\left(\frac{q}{d}\right), \tag{2.16}$$

where Ramanujan's trigonometrical sum is originally defined by

$$c_q(n)=\sum_{\substack{1\leqslant h\leqslant q\\ \text{g.c.d.}(h,q)=1}} \exp\{2\pi inh/q\}.$$

By (2.16) we have $|c_q(n)|\leqslant\min\{\varphi(q), \sum_{d\mid n} d\}$ and further [1, pp. 25–26]

$$D(n)=\varphi(n)=\sum_{d\mid n} d\mu\left(\frac{n}{d}\right);$$

$$F(n)=\sum_{d\mid n}\mu(d)=\begin{cases}1 & (n=1)\\ 0 & (n>1),\end{cases}$$

where $\varphi(\cdot)$ denotes Euler's totient function. Hence (2.4) reduces to

$$\begin{aligned}&\frac{\pi^{\nu}t^{\nu/2}}{2\Gamma(\nu+1)}\varphi(q)+\sum_{n\geqslant 1} c_q(n)\,n^{-\nu}J_\nu(2\pi n\sqrt{t})\\ &=\frac{\pi^{\nu-1/2}}{2\Gamma(\nu+\frac{1}{2})}t^{(1/2)(\nu-1)}F(q)\\ &+\frac{\pi^{\nu-1/2}t^{-\nu/2}}{\Gamma(\nu+\frac{1}{2})}\sum_{d\mid q}\mu(d)\sum_{\substack{1\leqslant m<d\sqrt{t}\\ m\in\mathbb{N}}}\left\{t-\left(\frac{m}{d}\right)^2\right\}^{\nu-1/2}.\end{aligned} \tag{2.17}$$

Now observe that, e.g., (i) $c_1(n)=\varphi(1)=F(1)=1$ and (ii) $c_2(n)=(-1)^n$, $\varphi(2)=1$, $F(2)=0$. These special cases of (2.17) are due to N. Nielsen [10, p. 336; 13, Sect. 19.41], E. C. Titchmarsh [12, pp. 65, 274], G. Doetsch [3, Part II, p. 240], and L. Lorch and P. Szego [9, p. 50] and have been applied by A. Hautot [6] in mathematical physics.

3. Hankel Transforms

Theorem A represents a linear functional relation for the Hankel kernel. We transfer this result to the space of Hankel transforms by Erdelyi's method [4, Sect. 2]. But observe that there is no general theorem concerning the term-wise application of the Hankel transform which covers all possible cases. With respect to our problem no serious difficulties arise in the case of finite Hankel transforms. Thus we can prove the general

THEOREM B. *Let $H_\nu f$ be a finite Hankel transform* (1.2), *i.e.*, $I=(\alpha,\beta)\subset\mathbb{R}^+$, $0=\alpha<\beta<+\infty$ *and* $f\equiv 0$ *in* $\mathbb{R}^+\backslash I$. *Then for* $q\in\mathbb{N}$ *and* $\operatorname{Re}\nu>\frac{1}{2}$

$$\frac{\pi^\nu}{2\Gamma(\nu+1)}D(q)\,S_f^{(\nu)}+\sum_{n\geq 1}s_q(n)\,n^{-\nu}(H_\nu f)(\pi^2n^2)$$
$$=\frac{\pi^{\nu-1/2}}{2\Gamma(\nu+\frac{1}{2})}F(q)\,S_f^{(\nu-1)}$$
$$+\frac{\pi^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})}\sum_{d\mid q}g(d)\,h\left(\frac{q}{d}\right)d^{-1}\sum_{\substack{1\leq m<d\sqrt{t}\\ m\in\mathbb{N}}}T_f^{(\nu)}\left(\frac{m}{d^2}\right),\tag{3.1}$$

where for $\xi\in I$, $t^{-\nu/2}f(t)(t-\xi)^{\nu-1/2}\in L^1(I)\cap C(I)$ *and*

$$S_f^{(\nu)}:=\int_I t^{\nu/2}f(t)\,dt,\qquad T_f^{(\nu)}(\xi):=\int_{\max(\alpha,\,\xi)}^{\beta}t^{-\nu/2}f(t)(t-\xi)^{\nu-1/2}\,dt.\tag{3.2}$$

Proof. By (1.2) we have for $n\in\mathbb{N}$, $I=(\alpha,\beta)$

$$(H_\nu f)(\pi^2n^2)=\int_\alpha^\beta J_\nu(2\pi n\sqrt{t})\,f(t)\,dt.\tag{3.3}$$

Recall the well-known asymptotic equality

$$J_\nu(z)\sim\sqrt{\frac{2}{\pi}}\,z^{-1/2}\cos\left\{z-\frac{1}{2}\nu\pi-\frac{\pi}{4}\right\}$$

($\operatorname{Re}\nu\geq 0$, $z\in\mathbb{R}^+$, $z\to+\infty$). Thus by (2.3) the series

$$\sum_{n\geq 1}s_q(n)\,n^{-\nu}J_\nu(2\pi n\sqrt{t})$$

converges absolutely and uniformly for $\operatorname{Re}\nu>\frac{1}{2}$ in any finite t-interval $I\subset\mathbb{R}^+$. Thus term-wise integration is permissible and by Theorem A we obtain with (3.3) and (3.2)

$$\frac{\pi^\nu}{2\Gamma(\nu+1)}D(q)\,S_f^{(\nu)}+\sum_{n\geq 1}s_q(n)\,n^{-\nu}(H_\nu f)(\pi^2n^2)$$
$$=\frac{\pi^{\nu-1/2}}{2\Gamma(\nu+\frac{1}{2})}F(q)\,S_f^{(\nu-1)}+\frac{\pi^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})}$$
$$\times\sum_{d\mid q}g(d)\,h\left(\frac{q}{d}\right)d^{-1}\int_\alpha^\beta t^{-\nu/2}f(t)\left[\sum_{\substack{1\leq m<d\sqrt{t}\\ m\in\mathbb{N}}}\left\{t-\left(\frac{m}{d}\right)^2\right\}^{\nu-1/2}\right]dt.$$

Since the series on the right-hand side is finite term-wise integration is clearly allowed. By (3.2) we obtain the required result.

Remark. In the important limiting case $\alpha=0$, $\beta=+\infty$, Theorem B becomes

$$\begin{aligned}
&\frac{\pi^{\nu}}{2\Gamma(\nu+1)} D(q)\, S_f^{(\nu)} + \sum_{n\geq 1} s_q(n)\, n^{-\nu}(H_\nu f)(\pi^2 n^2) \\
&\quad = \frac{\pi^{\nu-1/2}}{2\Gamma(\nu+\frac{1}{2})} F(q)\, S_f^{(\nu-1)} + \frac{\pi^{\nu-1/2}}{\Gamma(\nu+\frac{1}{2})} \sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1} \\
&\quad \times \sum_{m\geq 1} \int_{(m/d)^2}^{\infty} t^{-\nu/2} f(t) \left\{ t - \left(\frac{m}{d}\right)^2 \right\}^{\nu-1/2} dt, \qquad (3.4)
\end{aligned}$$

provided the allowance of the term-wise integrations and the existence of the integrals (3.2), which has to be checked in every special case.

4. A Generalized J_ν-Formula

We here consider a special case of Theorem B in order to prove a direct generalisation of Theorem A.

Theorem C. *Let $q\in\mathbb{N}$, $0<\beta<\infty$. Then for $\varepsilon>0$ and* Re $\nu>\frac{1}{2}$

$$\begin{aligned}
&\beta^{\nu} \sum_{n\geq 1} s_q(n)\left(\sqrt{\beta((n^2\pi^2+\varepsilon))}\right)^{-\nu} J_\nu\left(2\sqrt{\beta(n^2\pi^2+\varepsilon)}\right) \\
&\quad = \psi_1(q, \beta, \varepsilon, \nu) + \psi_2(q, \beta, \varepsilon, \nu) \qquad (4.1)
\end{aligned}$$

with

$$\begin{aligned}
\psi_1 :=& \frac{\beta^{\nu-1/2}}{2\sqrt{\pi}} (\sqrt{\varepsilon\beta})^{-(\nu-1/2)} J_{\nu-1/2}(2\sqrt{\varepsilon\beta})\, F(q) \\
&- \frac{\beta^{\nu}}{2} D(q) (\sqrt{\varepsilon\beta})^{-\nu} J_\nu(2\sqrt{\varepsilon\beta}) \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
\psi_2 :=& \frac{\varepsilon^{1/2-\nu}}{\sqrt{\pi}} \sum_{d\mid q} g(d)\, h\left(\frac{q}{d}\right) d^{-1} \sum_{\substack{1\leq m<d\sqrt{\beta} \\ m\in\mathbb{N}}} \left(\sqrt{\varepsilon\left(\beta-\left(\frac{m}{d}\right)^2\right)}\right)^{\nu-1/2} \\
&\times J_{\nu-1/2}\left(2\sqrt{\varepsilon\left(\beta-\left(\frac{m}{d}\right)^2\right)}\right). \qquad (4.3)
\end{aligned}$$

Proof. As in [4, Sect. 4.] we apply Theorem B with $\alpha=0$, $0<\beta<\infty$ and

$$f(t)=t^{\nu/2}(\beta-t)^{\mu/2}\,J_\mu(2\sqrt{\varepsilon(\beta-t)})$$

(Re $\nu>\frac{1}{2}$, Re $\mu>0$). Then for Re $s>0$

$$(H_\nu f)(s)=\int_0^\beta t^{\nu/2}(\beta-t)^{\mu/2}\,J_\mu(2\sqrt{\varepsilon(\beta-t)})\,J_\nu(2\sqrt{st})\,dt. \tag{4.4}$$

Recall Sonine's second finite integral [13, p. 376]

$$\int_0^{\pi/2} J_\mu(u\sin\vartheta)J_\nu(v\cos\vartheta)\sin^{\mu+1}\vartheta\cos^{\nu+1}\vartheta\,d\vartheta$$

$$=\frac{u^\mu v^\nu J_{\mu+\nu+1}(\sqrt{u^2+v^2})}{(u^2+v^2)^{(1/2)(\mu+\nu+1)}} \tag{4.5}$$

(Re $\mu>-1$, Re $\nu>-1$; u, $v\in\mathbb{C}$). Thus the substitution $t=\beta\cos^2\vartheta$ $(0\leqslant\vartheta\leqslant\pi/2)$ in (4.4) leads by (4.5) to the Hankel transform

$$(H_\nu f)(s)=\varepsilon^{\mu/2}s^{\nu/2}\left(\frac{\beta}{s+\varepsilon}\right)^{(1/2)(\mu+\nu+1)} J_{\mu+\nu+1}(2\sqrt{\beta(s+\varepsilon)}). \tag{4.6}$$

We have to evaluate the integrals in (3.2). Using Sonine's first finite integral [13, p. 373]

$$J_{\mu+\nu+1}(z)=\frac{z^{\nu+1}}{2^\nu\Gamma(\nu+1)}\int_0^{\pi/2}\cos^{2\nu+1}\vartheta\sin^{\mu+1}\vartheta\,J_\mu(z\sin\vartheta)\,d\vartheta \tag{4.7}$$

(Re $\mu>-1$, Re $\nu>-1$, $z\in\mathbb{C}$) we are led to the following fractional integrals of Erdelyi–Kober type (see, e.g., [5]),

$$\text{(i)}\quad S_f^{(\nu)}=\int_0^\beta t^\nu(\beta-t)^{\mu/2}\,J_\mu(2\sqrt{\varepsilon(\beta-t)})\,dt.$$

Substitute $t=\beta\cos^2\vartheta$ $(0\leqslant\vartheta\leqslant\pi/2)$. Then

$$S_f^{(\nu)}=2\beta^{\mu/2+\nu+1}\int_0^{\pi/2}\cos^{2\nu+1}\vartheta\sin^{\mu+1}\vartheta J_\mu(2\sqrt{\varepsilon\beta}\sin\vartheta)\,d\vartheta$$

and by (4.7) we obtain

$$S_f^{(\nu)}=2^{\nu+1}\beta^{\mu/2+\nu+1}\Gamma(\nu+1)(2\sqrt{\varepsilon\beta})^{-\nu-1}\,J_{\mu+\nu+1}(2\sqrt{\varepsilon\beta}), \tag{4.8}$$

while

$$S_f^{(\nu-1)} = \int_0^\beta t^{\nu-1/2}(\beta-t)^{\mu/2} J_\mu(2\sqrt{\varepsilon(\beta-t)})\,dt$$
$$= 2^{\nu+1/2}\beta^{\mu/2+\nu+1/2}\Gamma(\nu+\tfrac{1}{2})(2\sqrt{\varepsilon\beta})^{-\nu-1/2} J_{\mu+\nu+1/2}(2\sqrt{\varepsilon\beta}). \quad (4.9)$$

$$\text{(ii)}\quad T_f(\xi) = \int_\xi^\beta (\beta-t)^{\mu/2}(t-\xi)^{\nu-1/2} J_\mu(2\sqrt{\varepsilon(\beta-t)})\,dt.$$

Substitute $t=(\beta-\xi)\cos^2\vartheta+\xi$ $(0\leqslant\vartheta\leqslant\pi/2)$. Then

$$T_f^{(\nu)}(\xi) = 2(\beta-\xi)^{\mu/2+\nu+1/2}$$
$$\times \int_0^{\pi/2} J_\mu(2\sqrt{\varepsilon(\beta-\xi)}\sin\vartheta)\sin^{\mu+1}\vartheta\cos^{2\nu}\vartheta\,d\vartheta$$

and by (4.7) we obtain

$$T_f^{(\nu)}(\xi) = \varepsilon^{-\nu/2-1/4}(\beta-\xi)^{(\mu+\nu)/2+1/4} J_{\mu+\nu+1/2}(2\sqrt{\varepsilon(\beta-\xi)})\,\Gamma(\nu+1/2) \quad (4.10)$$

Inserting (4.6), (4.8), (4.9), and (4.10) into (3.1) of Theorem B we obtain for $\operatorname{Re}(\mu+\nu+1)>\frac{1}{2}$ by straightforward computation

$$\frac{\beta^{\mu+\nu+1}}{2} D(q)(\sqrt{\varepsilon\beta})^{-\mu-\nu-1} J_{\mu+\nu+1}(2\sqrt{\varepsilon\beta})$$
$$+\beta^{\mu+\nu+1}\sum_{n\geqslant 1} s_q(n)(\sqrt{\beta(n^2\pi^2+\varepsilon)})^{-\mu-\nu-1}$$
$$\times J_{\mu+\nu+1}(2\sqrt{\beta(n^2\pi^2+\varepsilon)})$$
$$=\frac{1}{2\sqrt{\pi}} F(q)\,\beta^{\mu+\nu+1/2}(\sqrt{\varepsilon\beta})^{-\mu-\nu-1/2} J_{\mu+\nu+1/2}(2\sqrt{\varepsilon\beta})$$
$$+\frac{1}{\sqrt{\pi}}\sum_{d\mid q} g(d)\,h\left(\frac{q}{d}\right) d^{-1} \sum_{\substack{1\leqslant m<d\sqrt{\beta}\\ m\in\mathbb{N}}} \left(\sqrt{\varepsilon\left(\beta-\left(\frac{m}{d}\right)^2\right)}\right)^{\mu+\nu+1/2}$$
$$\times J_{\mu+\nu+1/2}\left(2\sqrt{\varepsilon\left(\beta-\left(\frac{m}{d}\right)^2\right)}\right)\varepsilon^{-\mu-\nu-1/2}.$$

Hence the transformation $\mu+\nu+1\to\nu$ gives the required result for $\operatorname{Re}\nu>\frac{1}{2}$.

Remark. Except for constant factors Theorem A is the limiting case $\varepsilon\to 0+$ of Theorem C. This follows at once by the well-known relation

$$\lim_{z\to 0} z^{1/2-\nu} J_\nu(2z) = \{\Gamma(\nu+1)\}^{-1} \qquad (\operatorname{Re}\nu>-1).$$

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